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CONBINATORIAL PROPERTIES OF FINITE FULL TRANSFORMATION SEMIGROUPS

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Let X be the finite set $\{1, 2,\dots,n\}$ and let $T(X)$ be the semigroup (under composition of mappings from X into X. The symmetric group $G(X)$, consisting of all permutations of X, is a subgroup of $T(X)$, while the set $S_n = T(X)\backslash G(X)$ of all singular mappings from X into X is a subsemigroup of $T(X)$. We denote the *image* of α of S_n by $im\alpha$, i. e., $im\alpha = \{x\alpha \mid x \in X\}$, and define the *rank* of α to be $rank\alpha = |im\alpha|$. Let E be the set of idempotents of S_n. In [1], it has shown that S_n is generated by the $n(n-1)$ idempotents of rank n-1. Then there arise the following two problems :

Problem 1. Find the least integer k for which $E^k = S$.

Problem 2. For each $\alpha \in S_n$, find the least integer $k(\alpha)$ for which $\alpha \in E^{k(\alpha)}$.

Let E_1 be the set of idempotents of rank n-1 in S_n. Iwahori [3] and Howie [2] found the least integer $l(\alpha)$ for which $\alpha \in E_1^{l(\alpha)}$. By using this result, Howie [2] solved Problem 1, that is $k = [3(n-1)/2]$.

In this survey, we discuss on Problem 2. The proofs of the results here are not given. But to make the results understandable, we will give examples.

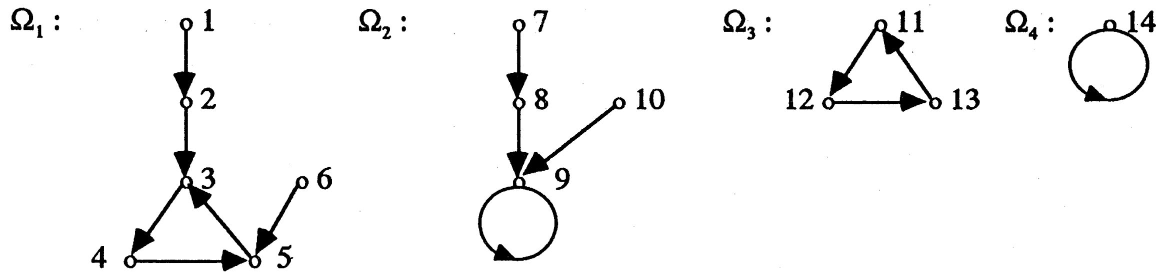
Let $\alpha \in S_n$. We define $fix\alpha = \{x \in X \mid x\alpha = x\}$, and an *orbit* of α to be an equivalence class under the equivalence $\omega = \{(x, y) \in X\times X \mid x\alpha^l = y\alpha^m$ for some $l, m \geq 0\}$. Then each orbit Ω of α has a kernel $K(\Omega)$ characterised by the property (for each x in Ω) $x \in K(\Omega)$ if and only if $x \in x\alpha^{-N}$ where $x\alpha^{-N} = \{y \in X \mid y\alpha^i = x$ for some $i \geq 1\}$. Then orbits classified into the following four types :

standard orbit : $|\Omega| > |K(\Omega)| > 1$
acyclic orbit : $|\Omega| > |K(\Omega)| = 1$
cyclic orbit : $|\Omega| = |K(\Omega)| > 1$
singleton orbit : $|\Omega| = |K(\Omega)| = 1$.

Example 1. Let $n = 14$ and let

$$\alpha = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 \\ 2 & 3 & 4 & 5 & 3 & 5 & 8 & 9 & 9 & 9 & 12 & 13 & 11 & 14 \end{pmatrix}$$

The orbits of α can be decicted as follows :



Then $|\Omega_1| = 6 > |K(\Omega_1)| = 3 > 1$, $|\Omega_2| = 4 > |K(\Omega_2)| = 1$, $|\Omega_3| = |K(\Omega_3)| = 3 > 1$, $|\Omega_4| = |K(\Omega_4)| = 1$, so that Ω_1 is standard, Ω_2 is acyclic, Ω_3 is cyclic and Ω_4 is singleton.

It is easy to see that $\alpha \in S_n$ is an idempotent if and only if $im\alpha = fix\alpha$. Thus we have that, if ε is an idempotent of rank n-1, then there exist a and b in X such that $a\varepsilon = b$ and $x\varepsilon = x$ if $a \neq b$. We write $\varepsilon = \begin{pmatrix} a \\ b \end{pmatrix}$. For example, $\begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 4 & 3 & 4 \end{pmatrix} = \begin{pmatrix} 2 \\ 4 \end{pmatrix}$.

Let $\alpha \in S_n$. Then the number of cyclic orbits of α is denoted by $c(\alpha)$. We define the *gravity* of α to be $g(\alpha) = n - |fix\alpha| + c(\alpha)$, and the *defect* of α to be $d(\alpha) = n - rank\alpha$.

THEOREM 1. (Nobuko Iwahori [3] and J. M. Howie [2])
Let S_n be the semigroup of all singular mappings from X into X where X is the finite set $\{1, 2, \ldots, n\}$ and let E_1 be the set of idempotents of defect 1 (rank n-1) in S_n. For each $\alpha \in S_n$ the least $l(\alpha)$ for which $\alpha \in E^{l(\alpha)}$ is $g(\alpha)$, where $g(\alpha)$ is the gravity of α.

We state the outline of the proof of Theorem 1 by using the α in Example 1. In this case, $|fix\alpha| = 2$ and $c(\alpha) = 1$, so that $g(\alpha) = 14 - 2 + 1 = 13$. For Ω_1, take $x \in \Omega_1$ such that $x \notin K(\Omega_1)$ and $x\alpha \in K(\Omega_1)$, say $x = 6$, and take $y \in K(\Omega_1)$ such that $x\alpha = y\alpha$, i. e., $x = 4$. Then

$$\Omega_1 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 3 & 4 & 5 & 3 & 5 \end{pmatrix} = \begin{pmatrix} 4 \\ 6 \end{pmatrix}\begin{pmatrix} 3 \\ 4 \end{pmatrix}\begin{pmatrix} 5 \\ 3 \end{pmatrix}\begin{pmatrix} 6 \\ 5 \end{pmatrix}\begin{pmatrix} 2 \\ 3 \end{pmatrix}\begin{pmatrix} 1 \\ 2 \end{pmatrix}.$$

For Ω_2, $\Omega_2 = \begin{pmatrix} 7 & 8 & 9 & 10 \\ 8 & 9 & 9 & 9 \end{pmatrix} = \begin{pmatrix} 8 \\ 9 \end{pmatrix}\begin{pmatrix} 7 \\ 8 \end{pmatrix}\begin{pmatrix} 10 \\ 9 \end{pmatrix}$.

For Ω_3, take $x \in X \setminus im\alpha$, say $x = 1$. Then

$$\Omega_3 = \begin{pmatrix} 11 & 12 & 13 \\ 12 & 13 & 11 \end{pmatrix} = \begin{pmatrix} 11 \\ 1 \end{pmatrix}\begin{pmatrix} 13 \\ 11 \end{pmatrix}\begin{pmatrix} 12 \\ 13 \end{pmatrix}\begin{pmatrix} 1 \\ 12 \end{pmatrix}.$$

We obtain $\alpha = \begin{pmatrix} 4 \\ 6 \end{pmatrix}\begin{pmatrix} 3 \\ 4 \end{pmatrix}\begin{pmatrix} 5 \\ 3 \end{pmatrix}\begin{pmatrix} 6 \\ 5 \end{pmatrix}\begin{pmatrix} 2 \\ 3 \end{pmatrix}\begin{pmatrix} 1 \\ 2 \end{pmatrix}\begin{pmatrix} 8 \\ 9 \end{pmatrix}\begin{pmatrix} 7 \\ 8 \end{pmatrix}\begin{pmatrix} 10 \\ 9 \end{pmatrix}\begin{pmatrix} 11 \\ 1 \end{pmatrix}\begin{pmatrix} 13 \\ 11 \end{pmatrix}\begin{pmatrix} 12 \\ 13 \end{pmatrix}\begin{pmatrix} 1 \\ 12 \end{pmatrix}$.

Let $a_1, \ldots, a_k$ be distinct elements in X, and let $b_1, \ldots, b_k$ be elements (not necessarily distinct) in X such that $\{a_1, \ldots, a_k\} \cap \{b_1, \ldots, b_k\} = \emptyset$. Then the semigroup generated by the idempotents $\begin{pmatrix} a_1 \\ b_1 \end{pmatrix}, \ldots, \begin{pmatrix} a_k \\ b_k \end{pmatrix}$ is a semilattice of order 2^{k-1} in which the rank of each element is greater than $n - k - 1$. We write $\begin{pmatrix} a_1 \\ b_1 \end{pmatrix} \ldots \begin{pmatrix} a_k \\ b_k \end{pmatrix} = \begin{pmatrix} a_1 & \ldots & a_k \\ b_1 & \ldots & a_k \end{pmatrix}$.

Then $\begin{pmatrix} a_1 & \ldots & a_k \\ b_1 & \ldots & a_k \end{pmatrix}$ is an idempotent of defect k (rank $n - k$).

Conversely, an idempotent of defect k can be written in the above form.

For α, $\beta \in S_n$, it is easy to see that $rank(\alpha\beta) \leq rank\alpha$ and $rank(\alpha b) \leq rank\beta$, so that $d(\alpha) \leq d(\alpha\beta)$ and $d(\beta) \leq d(\alpha\beta)$.

LEMMA 1. *Let $\alpha \in S_n$. Then $g(\alpha)/d(\alpha) \leq k(\alpha)$, where $k(\alpha)$ means that of Problem 2.*
Proof. Let $\alpha = \varepsilon_1\varepsilon_2 \ldots \varepsilon_{k(\alpha)}$, where each ε_i ($i = 1, 2, \ldots, k(\alpha)$) is an idempotent with $d(\varepsilon_i) \leq d(\alpha)$. Let $d(\varepsilon_i) = d_i$. Since an idempotent of defect d_i is a product of d_i idempotents of defect 1, α is a product of $d_1 + \ldots + d_{k(\alpha)}$ idempotents of defect 1. By Theorem 1, $g(\alpha) \leq d_1 + \ldots + d_{k(\alpha)} \leq d(\alpha)k(\alpha)$. Thus $g(\alpha)/d(\alpha) \leq k(\alpha)$.

LEMMA 2. *Let $a, b, c \in X$. Then*
(1) $\begin{pmatrix} a \\ b \end{pmatrix}\begin{pmatrix} a \\ c \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix}$, *where $a \neq b$, $a \neq c$.*

(2) $\begin{pmatrix} a \\ b \end{pmatrix}\begin{pmatrix} b \\ c \end{pmatrix} = \begin{pmatrix} a & b \\ c & c \end{pmatrix} = \begin{pmatrix} b \\ a \end{pmatrix}\begin{pmatrix} a \\ c \end{pmatrix}$, *where $a \neq b$, $b \neq c$, $a \neq c$.*

We introduce a new notation to be more easily visible.
We write $\begin{pmatrix} a \\ b \end{pmatrix} = (b \leftarrow a)$, $\begin{pmatrix} b \\ a \end{pmatrix}\begin{pmatrix} c \\ b \end{pmatrix} = (a \leftarrow b)(b \leftarrow c) = (a \leftarrow b \leftarrow c)$
and $\begin{pmatrix} a & c \\ b & d \end{pmatrix} = \begin{cases} (b \leftarrow a) \\ (d \leftarrow c). \end{cases}$

LEMMA 3. *Let $a_1, \ldots, a_k, b_1, \ldots, b_m$ be distinct elements in X, and let $c \in X$ with $c \neq a_k$, $c \neq a_{k-1}$, $c \neq b_m$. Then*

$$(c \leftarrow a_k \leftarrow \ldots \leftarrow a_i \leftarrow \ldots \leftarrow a_1)(a_i \leftarrow b_m \leftarrow \ldots \leftarrow b_1)$$
$$= \begin{cases} (c \leftarrow a_k \leftarrow \ldots \leftarrow a_i \leftarrow \ldots \leftarrow a_1). \\ (a_{i-1} \leftarrow b_m \leftarrow \ldots \leftarrow b_1) \end{cases}$$

We suggest a proof of Lemma 3 by using the following example.

Example 2. $(4 \leftarrow 3 \leftarrow 2 \leftarrow 1)(3 \leftarrow 5 \leftarrow 6) = \begin{pmatrix} 3 \\ 4 \end{pmatrix}\begin{pmatrix} 2 \\ 3 \end{pmatrix}\begin{pmatrix} 1 \\ 2 \end{pmatrix}\begin{pmatrix} 5 \\ 3 \end{pmatrix}\begin{pmatrix} 6 \\ 5 \end{pmatrix}$

$$= \binom{3}{4}\begin{pmatrix} 2 & 5 \\ 3 & 3 \end{pmatrix}\binom{1}{2}\binom{6}{5} = \binom{3}{4}\binom{5}{2}\binom{2}{3}\binom{1}{2}\binom{6}{5}$$

$$= \left\{ \begin{array}{l} (4 \leftarrow 3 \leftarrow 2 \leftarrow 1) \\ (2 \leftarrow 5 \leftarrow 6) \end{array} \right. .$$

Example 3. Let $\beta = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\ 2 & 3 & 4 & 5 & 3 & 5 & 8 & 9 & 9 & 9 \end{pmatrix}$.

By the previous result of α in Example 1, we have

$$\beta = \binom{4}{6}\binom{3}{4}\binom{5}{3}\binom{6}{5}\binom{2}{3}\binom{1}{2}\binom{8}{9}\binom{7}{8}\binom{10}{9}$$

$$= \left\{ \begin{array}{l} (6 \leftarrow 4 \leftarrow 3 \leftarrow 5 \leftarrow 6)(3 \leftarrow 2 \leftarrow 1) \\ (9 \leftarrow 8 \leftarrow 7) \\ (9 \leftarrow 10) \end{array} \right. = \left\{ \begin{array}{l} (6 \leftarrow 4 \leftarrow 3 \leftarrow 5 \leftarrow 6) \\ (5 \leftarrow 2 \leftarrow 1) \\ (9 \leftarrow 8 \leftarrow 7) \\ (9 \leftarrow 10) \end{array} \right.$$

$$= \begin{pmatrix} 4 & 2 & 8 & 10 \\ 6 & 5 & 9 & 9 \end{pmatrix}\begin{pmatrix} 3 & 1 & 7 \\ 4 & 2 & 8 \end{pmatrix}\binom{5}{3}\binom{6}{5}.$$

Then we have that in the above expression of β the last member of each series $(\ldots \leftarrow \ldots)$ belongs to $X \setminus im\beta$ and they are mutually distinct.

The α of Example 1 can be expressed as follows :

$$\alpha = \left\{ \begin{array}{l} (6 \leftarrow 4 \leftarrow 7 \leftarrow 5 \leftarrow 6) \\ (5 \leftarrow 2 \leftarrow 1 \leftarrow 11 \leftarrow 13 \leftarrow 12 \leftarrow 1) \\ (9 \leftarrow 8 \leftarrow 7) \\ (9 \leftarrow 10) \end{array} \right.$$

Then the number of series in the above expression of α coincides with $d(\alpha)$ and the number of all arrows coincides with $g(\alpha)$.

Lemma 4. *Let $a_1, \ldots, a_m$ ($m \geq 3$) be distinct elements in X and let $\begin{pmatrix} a_m & b \\ c & d \end{pmatrix}$ be an idempotent of defect 2. Then*

$$\left\{ \begin{array}{l} (c \leftarrow a_m \leftarrow \ldots \leftarrow a_i \leftarrow \ldots \leftarrow a_1) \\ (d \leftarrow b) \end{array} \right. = \left\{ \begin{array}{l} (c \leftarrow a_m \leftarrow \ldots \leftarrow a_{i+1} \leftarrow b) \\ (d \leftarrow b \leftarrow a_i \leftarrow \ldots \leftarrow a_1) . \end{array} \right.$$

We also suggest a proof of Lemma 4 by using the following example.

Example 3. $\begin{cases}(5 \leftarrow 4 \leftarrow 3 \leftarrow 2 \leftarrow 1) \\ (6 \leftarrow 7)\end{cases} = \begin{pmatrix} 4 & 7 \\ 5 & 6 \end{pmatrix}\begin{pmatrix} 3 \\ 4 \end{pmatrix}\begin{pmatrix} 2 \\ 3 \end{pmatrix}\begin{pmatrix} 1 \\ 2 \end{pmatrix}$

$$= \begin{pmatrix} 4 & 7 \\ 5 & 6 \end{pmatrix}\begin{pmatrix} 7 \\ 2 \end{pmatrix}\begin{pmatrix} 3 \\ 4 \end{pmatrix}\begin{pmatrix} 2 \\ 3 \end{pmatrix}\begin{pmatrix} 1 \\ 2 \end{pmatrix} \quad \text{(by (1) of Lemma 2)}$$

$$= \begin{pmatrix} 4 & 7 \\ 5 & 6 \end{pmatrix}\begin{pmatrix} 3 \\ 4 \end{pmatrix}\begin{pmatrix} 7 \\ 2 \end{pmatrix}\begin{pmatrix} 2 \\ 3 \end{pmatrix}\begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

$$= \begin{pmatrix} 4 & 7 \\ 5 & 6 \end{pmatrix}\begin{pmatrix} 3 \\ 4 \end{pmatrix}\begin{pmatrix} 2 \\ 7 \end{pmatrix}\begin{pmatrix} 7 \\ 3 \end{pmatrix}\begin{pmatrix} 1 \\ 2 \end{pmatrix} \quad \text{(by (2) of Lemma2)}$$

$$= \begin{pmatrix} 4 & 7 \\ 5 & 6 \end{pmatrix}\begin{pmatrix} 3 & 2 \\ 4 & 7 \end{pmatrix}\begin{pmatrix} 7 & 1 \\ 3 & 2 \end{pmatrix}$$

$$= \begin{cases}(5 \leftarrow 4 \leftarrow 3 \leftarrow 7) \\ (6 \leftarrow 7 \leftarrow 2 \leftarrow 1)\,.\end{cases}$$

The length of $(a_m \leftarrow \ \dots \ \leftarrow a_1)$ is the number of arrows in it. Lemma 4 shows that the length of $(c \leftarrow a_m \leftarrow \ \dots \ \leftarrow a_1)$ decreases by k and the length of $(d \leftarrow b)$ increases by $k + 1$.

Let $V_0 = \{v_1, v_2, \dots, v_d\}$ be a multi-set of positive integers ($d \geq 2$), where $v_1, \dots, v_d$ are not necessarily distinct. Let us subtract k from some v_i and add $k + 1$ to some v_j where k is a integer. Let $V_1 = \{v_1, \dots, v_i\text{-}k, \dots, v_j\text{+}k\text{+}1, \dots, v_d\}$. By repeating this procedure on V_1, we obtain a new multi-set V_2.

LEMMA 5. *Let $V_0 = \{v_1, v_2, \dots, v_d\}$ be a multi-set of positive integers ($d \geq 2$) with $v_1 + v_2 + \dots + v_d = g$. By suitable repeating of the above procedure, there exists V_t such that $\lceil g/d \rceil \leq maxV_t \leq \lceil g/d \rceil + 1$ and $maxV_t = \lceil g/d \rceil$ if $g \equiv 1 \pmod d$, where $\lceil x \rceil$ denotes the least integer m for which $m \geq x$.*

Example 5. Let $V_0 = \{1, 8, 26, 32, 54\}$. Then $V_1 = \{31, 8, 25, 32, 25\}$, $V_2 = \{31, 16, 26, 25, 25\}$, $V_3 = \{25, 23, 26, 25, 25\}$ and $V_4 = \{25, 25, 25, 25, 25\}$.

Let α be as in Example 1. Then $\alpha = \begin{cases}(6 \leftarrow 4 \leftarrow 3 \leftarrow 5 \leftarrow 6) \\ (5 \leftarrow 2 \leftarrow 1 \leftarrow 11 \leftarrow 13 \leftarrow 12 \leftarrow 1) \\ (9 \leftarrow 8 \leftarrow 7) \\ (9 \leftarrow 10)\ .\end{cases}$

Let V_0 be the multi-set of the lengths of the series in the above expression of α, i. e., $V_0 = \{4, 6, 2, 1\}$. By applying Lemma 5 to the expression of α, we have

$$\alpha = \begin{cases} (6 \leftarrow 4 \leftarrow 3 \leftarrow 5 \leftarrow 6) \\ (5 \leftarrow 2 \leftarrow 1 \leftarrow 11 \leftarrow 10) \\ (9 \leftarrow 8 \leftarrow 7) \\ (9 \leftarrow 10 \leftarrow 13 \leftarrow 12 \leftarrow 1) \end{cases} = \begin{pmatrix} 4 & 2 & 8 & 10 \\ 6 & 5 & 9 & 9 \end{pmatrix} \begin{pmatrix} 3 & 1 & 7 & 13 \\ 4 & 2 & 8 & 10 \end{pmatrix} \begin{pmatrix} 5 & 11 & 12 \\ 3 & 1 & 13 \end{pmatrix} \begin{pmatrix} 6 & 10 & 1 \\ 5 & 11 & 12 \end{pmatrix}.$$

In this case, $V_1 = \{4, 4, 2, 4\}$ and $max V_1 = 4 = \lceil 13/4 \rceil = \lceil g(\alpha)/d(\alpha) \rceil$. Thus we obtain :

THEOREM 2. *Let S_n be the semigroup of all singular mappings from X into X where $X = \{1, 2, ..., n\}$, and let E be the set of idempotents of S_n. For each $\alpha \in S_n$, let $k(\alpha)$ be the unique positive integer for which $\alpha \in E^{k(\alpha)}$, $\alpha \notin E^{k(\alpha)-1}$, and $g(\alpha)$ the gravity of α and $d(\alpha)$ the defect of α. Then $k(\alpha) = \lceil g(\alpha)/d(\alpha) \rceil$ or $\lceil g(\alpha)/d(\alpha) \rceil + 1$, and equals to $\lceil g(\alpha)/d(\alpha) \rceil$ if $g(\alpha) \equiv 1$ (mod $d(\alpha)$), where $\lceil x \rceil$ for any real number x denotes the least integer m for which $m \geq x$.*

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